Advancement to Candidacy Presentation

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Based on

"*Quantum Constraint Problems can be complete for BQP, QCMA, and more*" [arXiv:2101.08381]

(Quantum) Complexity Theory

- A *problem*: a collection of inputs and outputs we want to compute.
  - Each pair is an *instance*
  - Example:
    - Is this n-by-n matrix, *M*, positive definite? Gives set of (M, true/false) pairs.
    - Given this n-wire circuit, *C*, how many inputs will make the output “true”?
- Any particular instance can be solved in “constant” time
- Focus instead on how the difficult scales with problem *size*.
  - Scale with size of matrix M
  - Scale with number of wires

(Quantum) Complexity Theory

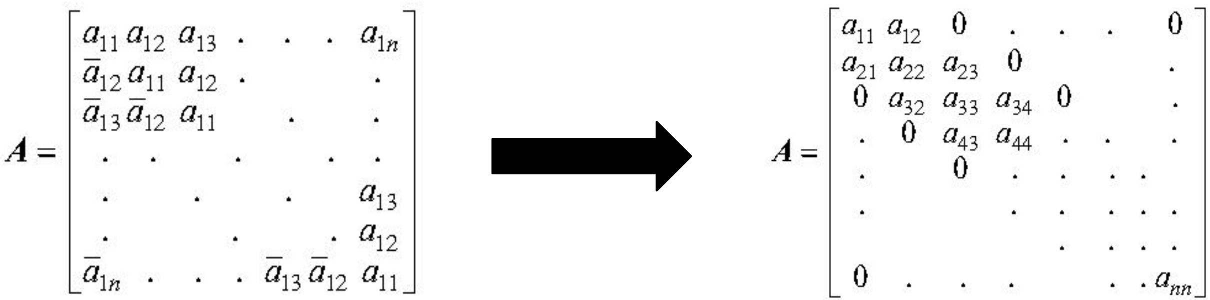
- Asymptotic resource usage (time, memory) to solve a class of problems
  - If we “scale up” the problem (more particles / larger matrices), how does the time needed change?
  - Some problems will go polynomially, others take exponentially longer and longer
  - Generally treat $O(n^2)$ vs $O(n^3)$ on similar footing: both reasonably doable.
  - $O(2^n)$ vs $O(3^n)$: both rapidly become intractable!
- Results in a sharp, *qualitative* notion of difficulty.
- … in turn leads to the discovery of many more efficient algorithms, or that no efficient algorithm will exist (and we should focus on heuristics)

(Quantum) Complexity Theory

- Examples from quantum complexity theory:
  - You can simulate a quantum computer with moderate memory (but exponentially much time)
  - Quantum computers can invert matrices in √(memory needed for regular computers)
  - Finding the ground state of gapless 1D Hamiltonians is “as hard as any quantum problem”
    - But easily solved for gapped
    - Case of O(1/n) gaps is still open

(Quantum) Complexity Theory

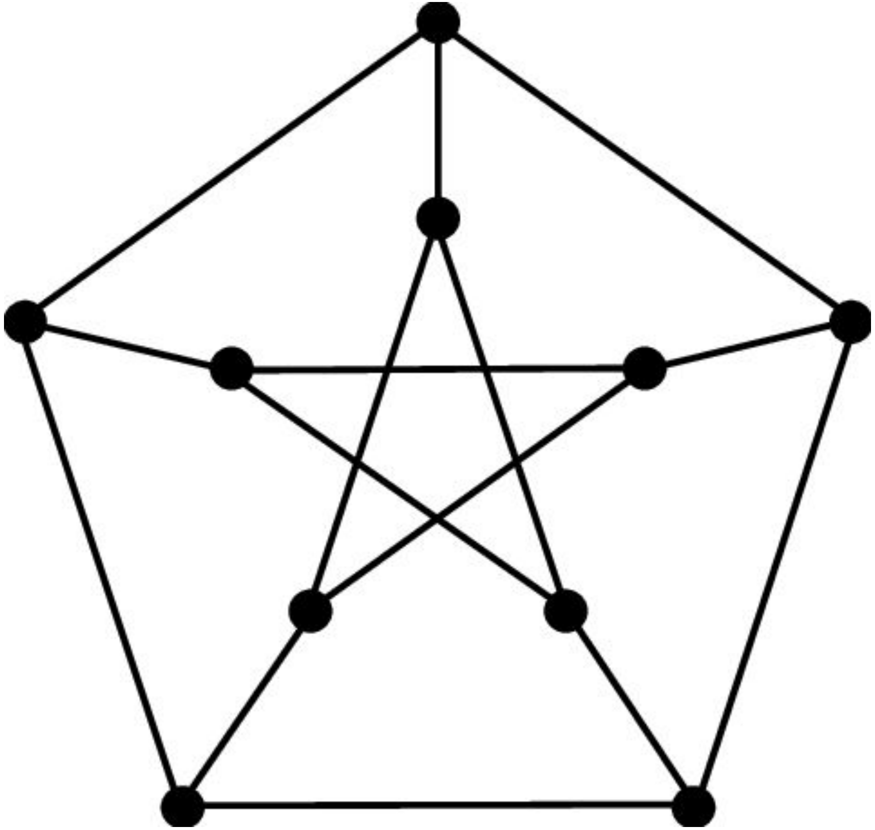
- Definition: Complexity classes
  - Equivalence classes of problems
  - Problem A ≤ Problem B if I can easily turn an A instance into a B instance
    - Problem A: Find the eigenvalues of a Hermitian matrix.
    - Simple algorithm: turn a Hermitian matrix into tridiagonal (sparse) matrix
    - Lets me focus on eigenvalue problem of tridiagonal matrices (Problem B).

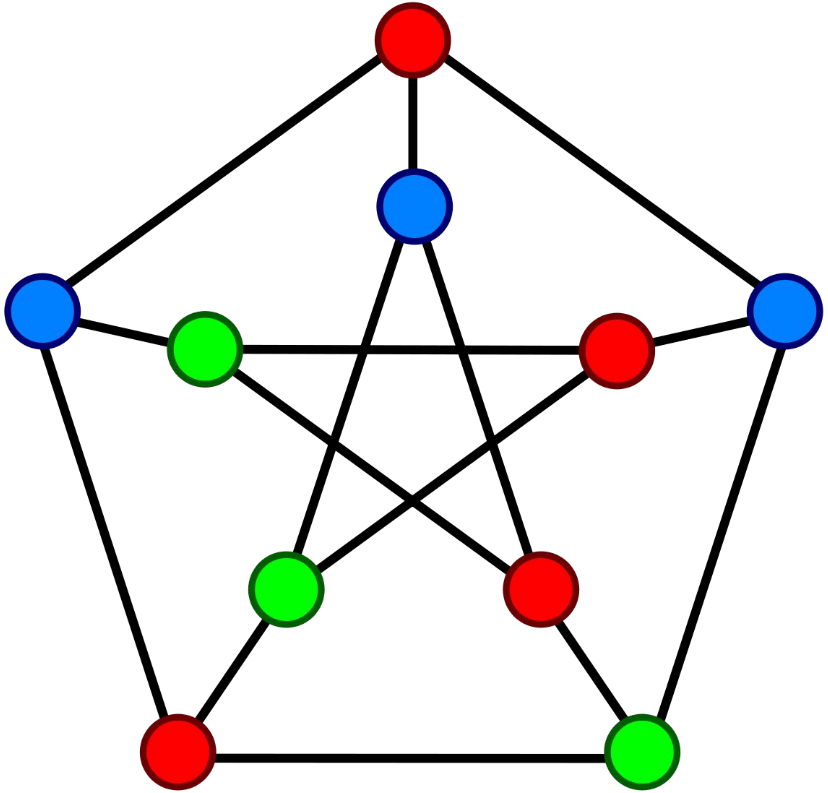


- Problem B ≤ Problem A (tridiagonal are a special case)
  - Conclude that the class A = class B. Not the same problem, but equal difficulty.

P vs. NP

- P: Easy to solve.
  - Can be solved in **P**olynomial time. Could be *O(n)* time or *O(n^5)* time, or anything else.
  - Example: Diagonalize an *n* by *n* matrix.
- NP: Easy to check.
  - **N**ondeterministic **P**olynomial time.
  - If you *guessed* the answer, you could check it very easily.
  - Find a solution to a system of *n* quadratic equations in *n* variables.
  - Color a network graph with three colors.





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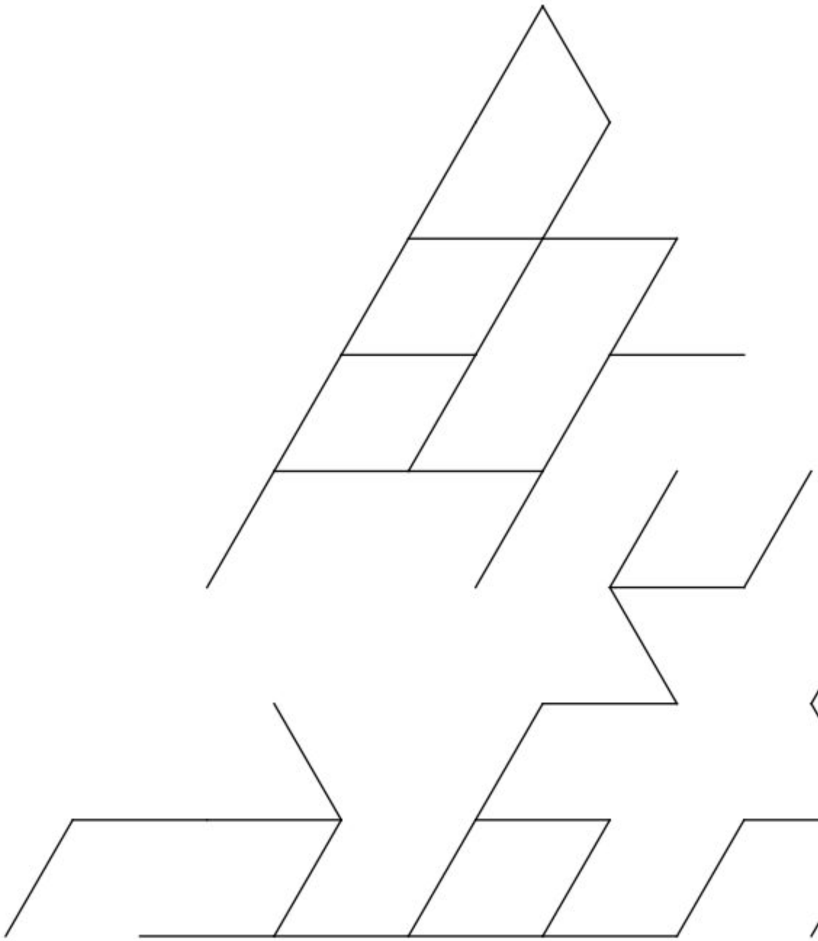
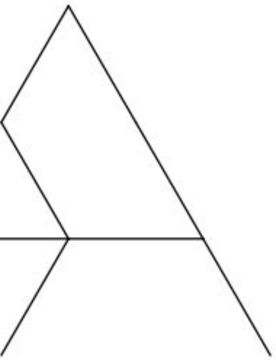
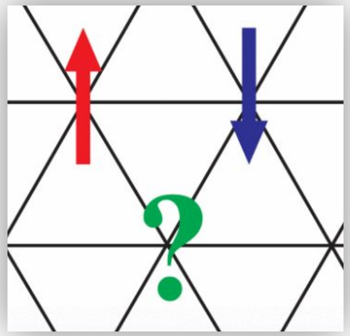
    - Absence of frustration in an Ising model (spin-½ vs spin-1)



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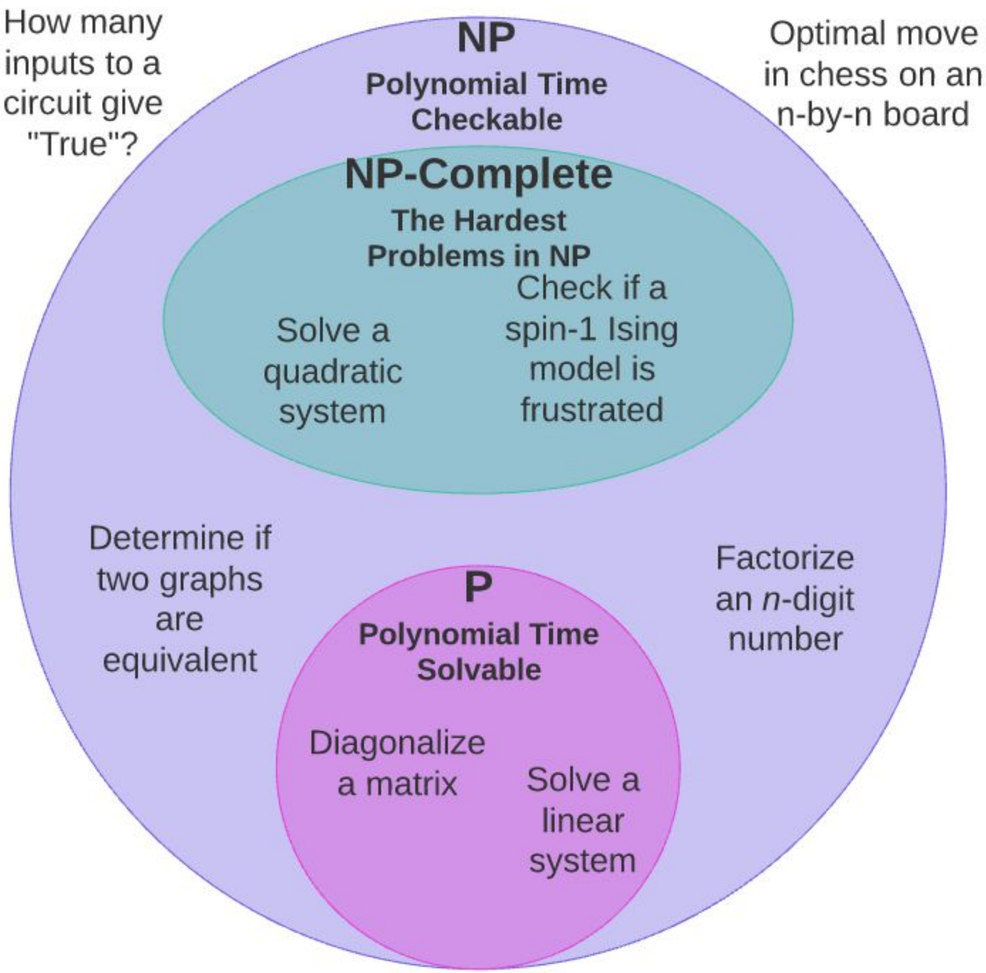
  - Find a solution to a system of *n* quadratic equations in *n* variables.
  - Color a network graph with three colors.
  - Absence of frustration in an Ising model (spin-½ vs spin-1)
- Could these be equal?
  - Probably not.
  - One of the great Millenium Prize problems in mathematics, $1M prize.


How many inputs to a circuit give "True"?
NP
Polynomial Time Checkable
Optimal move in chess on an n-by-n board
NP-Complete
The Hardest Problems in NP
Solve a quadratic system
Check if a spin-1 Ising model is frustrated
Determine if two graphs are equivalent
Factorize an *n*-digit number
P
Polynomial Time Solvable
Diagonalize a matrix
Solve a linear system

Constraint problems (classical)

- Variables chosen from some finite set (True/False; Red/Green/Blue)
- Relationships between fixed number of variables
  - v1 is true *or* v2 is false
  - At least one of (v3, v5, v10) is red
- Problem: is there an allowed assignment of variables?
- Sometimes the problem is very easy
  - Can follow a chain of implications and deduce an answer if there is one, or prove there isn't. Class P.
- Sometimes the problem is very hard
  - Too many options at each step. Have to resort to guessing and checking.
  - *...but at least it's still easy to check!*
  - Is **universal** for "problems that are easy to check"! (Cook, 1971) Forms the class NP-complete.

Constraint problems (classical)

- Can be viewed as minimizing an energy functional:
  - E(v1, v2) = 1 *if (v1 is false and v2 is true), else* 0.
  - E(v3, v5, v10) = 1 *if all of (v3,v5,v10) are not red, else* 0.
- Overall Hamiltonian is a sum of these interactions
- Question: Is there an E_{Tot}=0 state?

Classifications

- Outside of constraint problems, some are (believed to be) harder than P, but easier than NP
  - Example: integer factorization.
  - No know polynomial-time algorithm (harder than P)
  - Can be easily checked (NP is an upper bound on difficulty)
  - Despite much searching, does not seem to capture full NP difficulty
    - Constraint problems cannot be written in terms of factorization

- Constraint problems: always either easy (P) or maximally hard (NP)?
  - Called the “Dichotomy conjecture”, open for many years
  - Finally proved by Zhuk (2017)

Constraint problems (quantum)

- Variables are now qubits (or, generally, qudits)
- Form a Hamiltonian from a sum of local projectors
  - $H(v1,v2) = (1 + \sigma_{1,x}\sigma_{2,y})/2$
  - $H(v4,v5,v8) = 1 - |002\rangle\langle 002| - |12+\rangle\langle 12+|$
- Does this Hamiltonian have a zero-energy ground state?
  - *i.e.* Is this Hamiltonian frustration-free, *or* is the ground state energy larger than zero?
- Hard to find the answer. But given the ground state, easy to check.
  - Measure the provided ground state on each local projector. Positive chance to find violated term.

Constraint problems (quantum)

- Problems checkable given a quantum state: QMA
- Kitaev (2002) showed 5-local Hamiltonians on qubits are universal for QMA, that is, QMA-complete.
    - Since improved to 3-local Hamiltonians on qubits.
    - 2-local Hamiltonians have an efficient algorithm for determining frustration: in P.

Classifications - in the quantum setting

- Classical problems can still be realized as quantum constraint problems, so there are “P” and “NP” quantum problems.
- Kitaev showed that there are QMA (quantum NP) complete problems.
- In 2008, Bravyi & Terhal show that “stoquastic” frustration-free Hamiltonians are MA-complete.
    - Stoquastic: the off-diagonal elements of the operators are real and non-positive. These are Hamiltonians “with no sign problem”, and permit efficient Monte-Carlo methods in many settings.
    - MA-complete: the same as NP, but verification is allowed be probabilistic.
        - “Give me your proof, I’ll run many checks, and >80% of my checks should pass.”

- … but no evidence this list is complete.

Classifications - in the quantum setting

Question:

Is there a class of Hamiltonians that captures exactly the power of quantum computers? (BQP-complete)

- **BQP**: Problems with a quantum circuit that to solve them,
  - Correct at least 2/3 of the time
  - Polynomially much time
  - **B**ounded-error **Q**uantum **P**olynomial
- **BQP-complete**: Problems that are sufficiently flexible to capture all of BQP.
  - Simple example: "What is the output of this quantum circuit"
  - Approximating Jones polynomials of knots

Classifications - in the quantum setting

Question:

Is there a class of Hamiltonians whose ground states capture exactly the power of quantum computers? (BQP-complete)

→ Compare with classical case of “P”, the problems that are ‘as hard as running an arbitrary program’ on a classical computer.

- Have to be flexible enough to simulate a full quantum computer
- Have to be constrained enough that a quantum computer can systematically proceed through and check for frustration.
- BQP is most “naturally” about quantum circuits. Nothing about ground states of Hamiltonians!

New results

- Yes! There is a BQP-complete class of Hamiltonian problems.
- Precise statement: there is a fixed list of interactions $\{H_1, H_2, H_3, H_4, H_5\}$ such that applying these to any qubits in any configuration gives a total Hamiltonian **H** that…
    - …can be used to simulate an arbitrary quantum computer C: **H**(C) is frustration free iff C returns “1”
    - …can be solved on a quantum computer: linear-time algorithm to determine if **H** is frustration free or not.

New results

- Bonus: once this set of interactions was designed, offered straightforward modifications to get two more new classes.
  - QCMA: “Quantum Classical Merlin-Arthur”. Problems checkable by a quantum computer given a classical solution string of bits
    - Harder than BQP (needs a solution) but easier than QMA (it isn’t a *quantum* solution)
  - RP: “Randomized polynomial”. Problems checkable by a classical computer with a source of randomness.
    - A very “classic” complexity class. Very few complete problems, surprisingly!
    - ... and here, a complete problem, that uses quantum mechanics!

Classical difficulty levels

Of constraint problems

- P
- NP (Cook, 1971)

Quantum difficulty levels

Of constraint problems

- P
- NP
- MA (Bravyi, 2008)
- QMA (Kitaev, 2002)

Classical difficulty levels

Of constraint problems

- P
- NP (Cook, 1971)

Quantum difficulty levels

Of constraint problems

- P
- RP (new)
- NP
- MA (Bravyi, 2008)
- BQP (new)
- QCMA (new)
- QMA (Kitaev, 2002)

Classical difficulty levels

Of constraint problems

- P
- NP (Cook, 1971)

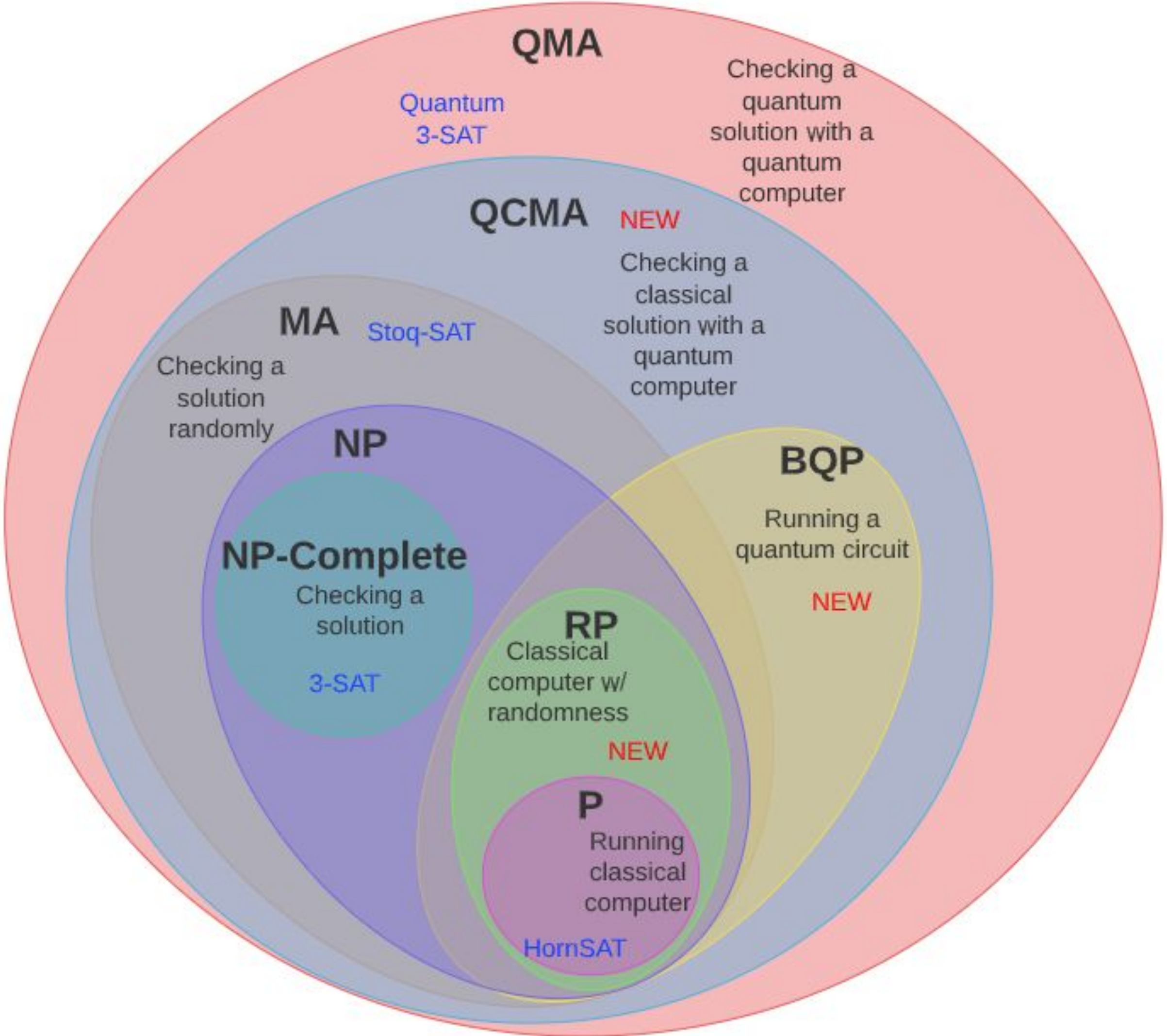
Known to be exhaustive.

Quantum difficulty levels

Of constraint problems

- P
- RP (new)
- NP
- MA (Bravyi, 2008)
- BQP (new)
- QCMA (new)
- QMA (Kitaev, 2002)

...maybe more to find?


QMA
Quantum 3-SAT
Checking a quantum solution with a quantum computer
QCMA
NEW
Checking a classical solution with a quantum computer
MA
Stoq-SAT
Checking a solution randomly
NP
NP-Complete
Checking a solution
3-SAT
BQP
Running a quantum circuit
NEW
RP
Classical computer w/ randomness
NEW
P
Running classical computer
HornSAT

Construction of a BQP-complete Hamiltonian

- Going to build a “dictionary” from circuits to Hamiltonians
- Circuit ➔ Hamiltonian:
  - Every circuit can be embedded in a Hamiltonian
  - Hamiltonian has low-energy state iff circuit outputs “1”
  - One such embedding was done with Kitaev’s clock construction.
- Hamiltonian ➔ Circuit:
  - Every Hamiltonian can be analyzed as a circuit
  - … or, if not a circuit exactly, then fragments of circuits, that can each be processed.

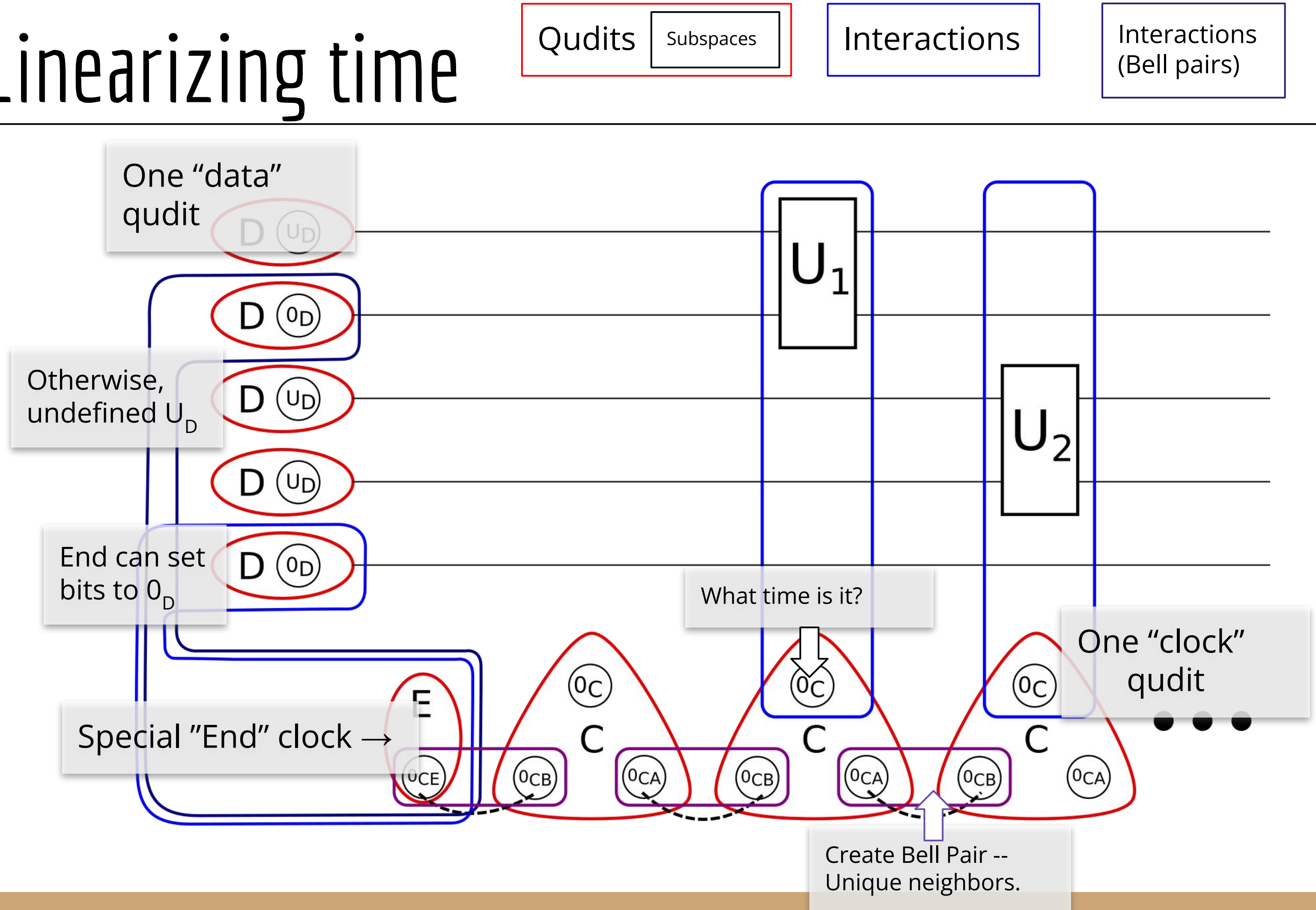
Construction of a BQP-complete Hamiltonian

- Idea: start with Kitaev's QMA-complete construction.
- Some qubits are "data", some are "time", overall the ground state is a "history" state (superposition of full computational history of circuit)
- Gives the ability to build any quantum circuit!
  - *But* that circuit can take any input: could be the "solution" quantum state. Can't have that!
  - *Also*, allows many configurations that are *not* quantum circuits:
    - Could use a "time" bit as "data" (what does this mean?), or have multiple "time" lines
    - Could couple leave "time" bits uncoupled
    - Input could be left blank or unusually constrained

Construction of a BQP-complete Hamiltonian

- Modify Kitaev's clock-circuits to be easily solvable.
- First, separate "data" and "clock" into separate states.
  - Qubits become qudits, with d=4: "data-0", "data-1", "clock-0", "clock-1".
- Penalize interactions that "don't look like circuits".
  - Local frustration appears.
  - Checker can quickly find these local problems and return "FRUSTRATED".
- Any absent constraints will make circuit trivially satisfiable
  - Failed to initialize the circuit correctly? Okay, we can put everything in an extra "dumb" state $|U\rangle$ that will satisfy everything else.

Linearizing time



Construction of a BQP-complete Hamiltonian

- Some “problems” include:
  - Time without the endpoints
    - All qubits end up in “undefined” state and work out trivially
    - Also applies to “circular time”
  - Multiple “timelines” acting on the same set of bits.
    - Choose gate set such that clock-data entanglement is guaranteed.
    - Bound the entanglement between data with each clock line.
    - Conclude that there is frustration.

Construction of a BQP-complete Hamiltonian

- When all the “fixes” are accounted for, interactions are:
  - 5-local
  - on 13-qudits
- Some reductions show this could be reduced to
  - 80-local interactions
  - On qubits
  - … definitely could do better!

Freebies

- QCMA (Quantum Classical Merlin-Arthur)
    - Like BQP or QMA, now instead of starting with |0⟩ or |psi⟩ we start with Z bitstring |k⟩
    - Start with BQP circuit
    - Alternate “End” clause now allows input to be 0 or 1
    - But makes a copy (in the Z basis) to an extra qubit so entanglement is undetectable
- RP (Randomized Polynomial-time)
    - Like BQP, but all of the operations are classical, and we have randomness
    - Initial states are can be set to |0⟩ or |+⟩
    - Only allowed gates are classical gates

But as a physicist...

... why should I care about classifying the *algorithms* that can solve different Hamiltonians?

- Really a statement about the types of entanglement present in different Hamiltonians
  - P or NP: “essentially classical” entanglement
  - BQP or QCMA: “efficiently preparable” entanglement
  - QMA: entanglement that is likely *not* efficiently preparable
- Might help us develop new algorithms for finding ground states

Future questions...

- Two other interesting complexity classes: StoqMA and TIM
  - No longer restricted to frustration-free (instead, “What is the ground-state energy?”)
  - StoqMA: stoquastic interactions. Should be “easier” in some senses
  - TIM: Transverse-field Ising Model. Relevant for DWave machines. Not obviously easier, but not shown to be universal either
- Looking for ways towards a classification
  - Dimitry Zhuk’s proof for classical problems centered on ‘polymorphisms’. Unlikely to carry over to quantum case nicely.
- More physically reasonable constructions for BQP, QCMA, RP.

Thank you!