ME140A - Midterm 2 - Open Book

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1 1D Fixed Points - 20%

Consider the system,

$$x'(t) = (x^2 + x)(x^2 - x - 2)(x^2 - 3x + 2)^2$$

Find all the fixed points. (You shouldn't need a calculator!) Classify them by their stability. Don't answer "indeterminate" – a particular *test* of stability may be indeterminate if the test isn't enough to tell, but I want you to tell me which are stable or not!

It may be helpful to note that when x is very large and positive, x' is positive, and when x is very large and negative, x' is also positive. This is because the leading term in the polynomial is $+x^8$.

2 Initial Value Problems - 40%

Consider the following system:

$$x'' = -x - 3x^3 - t^2$$

with the initial conditions $x(0) = 1$ and $x'(0) = 0$. This system has a zero, i.e. a point t_0 where $x(t_0) = 0$, with t_0 in the interval $[0.5, 1]$. Finding the points where a differential equation crosses certain thresholds is called *event detection*, and is often the metric of interest.

(a) Use two steps of the predictor-correct method with $h = 0.5$, to estimate $x(0.5)$ and $x(1.0)$.

(b) Drawing a line between your two points from part (a), estimate time t_0 where $x(t_0) = 0$.

(c) From your halfway point of part (a) where you have $x(0.5)$, instead of doing a second predictor-corrector step, do a step of the Runge-Kutta 3th order (RK3) method with $h = 0.5$ to estimate $x(1.0)$. As a reminder, the RK3 method is:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(t_n + h,\, y_n + 2hk_2 - hk_1)$$

$$y_{n+1} = y_n + \frac{h}{6}\,(k_1 + 4k_2 + k_3)$$

What is $x(1.0)$ with this alternate step?

(d) Compare your estimates of $x(1.0)$ from parts (d) and (a). How large is the error? Treating this error as your uncertainty in $x(1.0)$, what is your uncertainty in t_0? Explain why this would or wouldn't be a good be a uncertainty to report.

3 High-dimensional Fixed Points - 40%

(a) The system

$$a''(t) = -6 - 2a + a^2 + 3b$$

$$b'(t) = -2 + a + b - 2ab^2 - a'$$

has several fixed points. Find all of them. (Two of them will have neat numerical forms.) You're allowed to use Wolfram Alpha to solve equations.

(b) Write down the Jacobian of this system. (Symbolically, so in terms of the variables, not at a particular fixed point.)

(c) Remember that the trace is the sum of the diagonal of the Jacobian, and that it is equal to the sum of the eigenvalues. What can the trace be for a stable fixed point? What is the trace of this matrix? Which fixed points can you immediately check are *not* stable, based on just the trace?

(d) For a 3x3 matrix

$$\begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix}$$

the (real parts of the) eigenvalues are all negative if and only if:

$$p \leq 0$$

$$pt - qs \leq 0$$

$$ptx + quv + rsw - puw - qsx - rtv + \leq 0$$

Evaluate these numerically at all the fixed points you *didn't* rule out in part (c). What are all the stable fixed points of the system?