ME140A - Midterm 1 - Open Book

Alex Meiburg

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1 Integration - 50%

Consider the integral,

$$F = \int_{-1}^{3} x^4 - 2x^3 \, dx$$

Remember that Simpson's 1/3 rule is,

$$\frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

1.1) Compute the exact value of F using algebra.

1.2) Compute an approximate integral with Simpson's 1/3 rule. Use the *simple* rule, so, $n = 1$ and $h = 1.0$.

1.3) Compute an approximate integral with the composite Simpson's 1/3 rule, with $n = 2$ and $h = 0.5$.

1.4) Use Richardson extrapolation, together with your answers from (1.2) and (1.3), to get a more precise estimate. Note that Simpson's 1/3 rule has $O(1/n^4)$ error scaling.

1.5) Compare your answer for (1.1) and (1.4) and give an explanation.

2 Differentiation - 50%

You want to use a computer to find the derivative of $f(x) = \sin(x)$ at $x = 1$). The exact answer is, of course, just $\cos(1)$. But you're going to use a finite difference method.

Useful facts: $\sin(1) \approx 0.8414$, $\cos(1) \approx 0.5403$. The Taylor series for $\sin(x)$ at the point $x = 1.0$ is

$$\sin(1 + x) = \sin(1) + \cos(1)\,x + \frac{-\sin(1)}{2}\,x^2 + \frac{-\cos(1)}{6}\,x^3 + O(x^4)$$

Do not use a calculator for this problem.

2.1) Write down the formula for a *forward finite-difference* to compute $f'(1)$, with $h = 0.01$. (This uses $f(x + h)$ and $f(x)$.) Don't actually evaluate sin, I just want to see the expression.

2.2) Using the Taylor series for $\sin(1 + x)$, estimate how much error your forward difference will have in the derivative, at the given $h = 0.01$.

2.3) Write down the formula for a *central* finite-difference to compute $f'(1)$, with $h = 0.01$.

2.4) Using the Taylor series for $\sin(1 + x)$, estimate how much error your central difference will have in the derivative, at the given $h = 0.01$.

2.5) Assume your computer stores 15 decimal digits of precision. This means that, when you add or subtract two numbers x and y, you get an error about $10^{-15}x$ or $10^{-15}y$, whichever is bigger. Write down a expression for the error from the *central* finite-difference from (2.3), that depends on h. This expression should include both the precision error from your computer, and the error from the finite-difference method itself. *There will be some range of answers will be accepted, because of different ways to estimate the rounding error.*

2.6) Using your error formula from (2.5), figure out an optimal value of h to minimize your total error. *Any answer within a factor of 4 of the optimum will be accepted, because of different ways to estimate the rounding error in (2.5).*