ME140A - Homework 4 - Solutions

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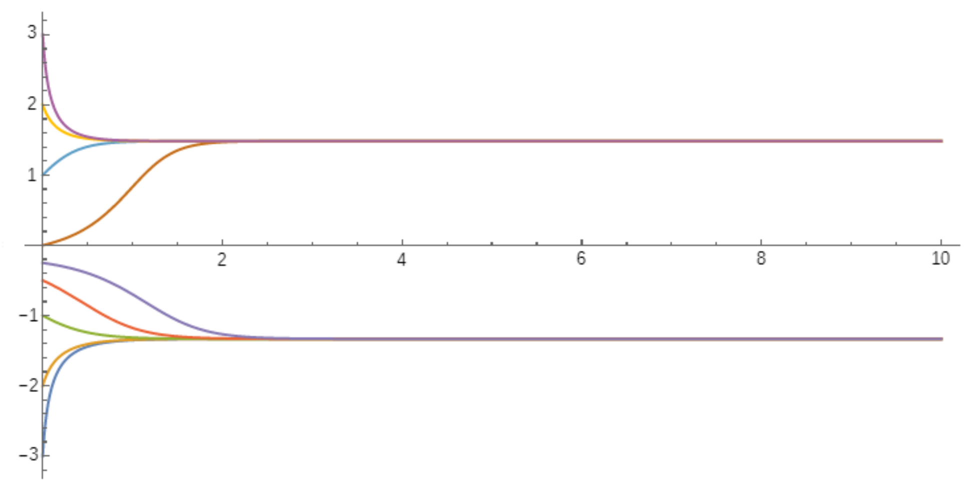
1 1D Fixed Points

(a) Solve the following time independent system:

$$x'(t) = 0.3 - x^3 + 2x$$

Plot the solutions with the following initial values for $x(0)$: -3, -2, -1, -0.5, -0.25, 0, 1, 2, and 3. Show at least the behavior from $t = 0$ up to $t = 10$. Example plot:



(b) Identify all three fixed points in the above equation. How does this match up with your observed behavior? Solving $0.3 - x^3 + 2x = 0$ gives fixed points:

$$x = -1.33222, \quad x = -0.151747, \quad x = 1.48397$$

We see the -1.33 and 1.48 fixed points on the plot, as where the solutions converge. The -0.15 might be a bit more surprising, but we can see that the curves are flat near $x = -0.25$ (the "middle" of the curves, where the split at early t).

(c) Classify the points from part (b) as stable or unstable. How does this match up with your observed behavior? At the three given points, the derivatives are

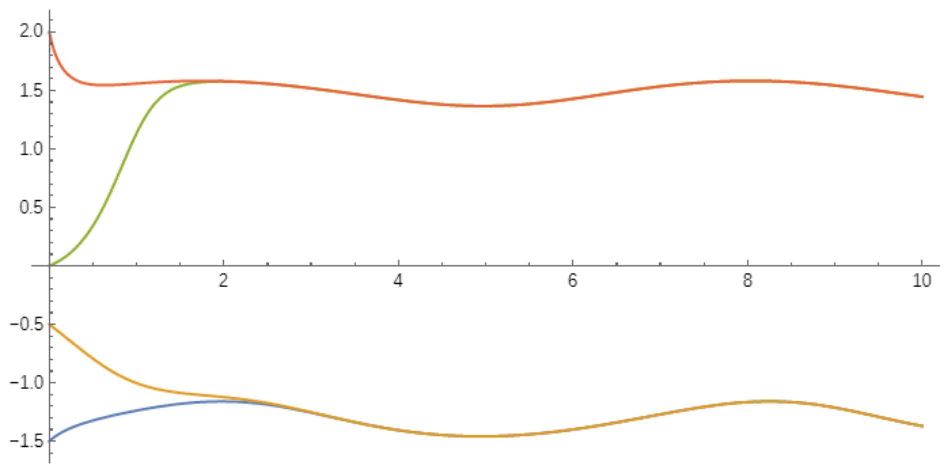
$$-3.32444, \quad 1.93092, \quad -4.60648$$

so they are stable, unstable, and stable, respectively. This makes sense: solutions converge to stable points, and diverge from unstable points.

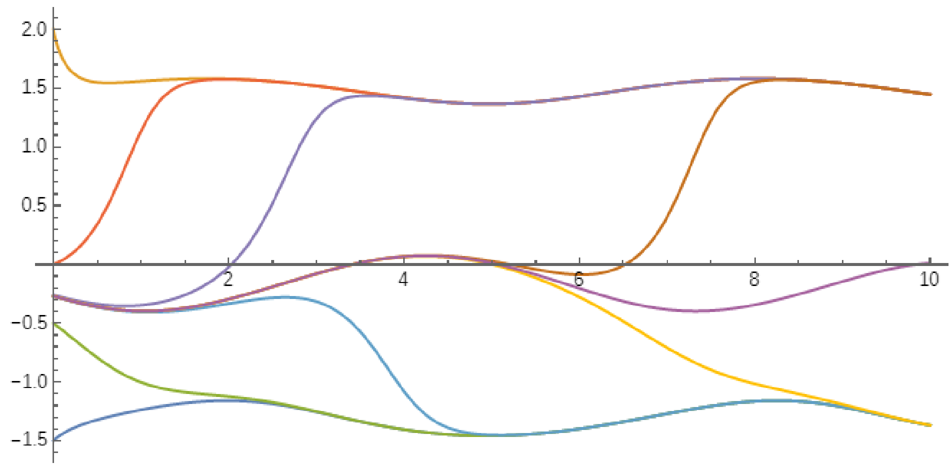
(d) Now add a *time-dependent* perturbation to the system:

$$x'(t) = 0.3 - x^3 + 2 * x + 0.5 \sin(t)$$

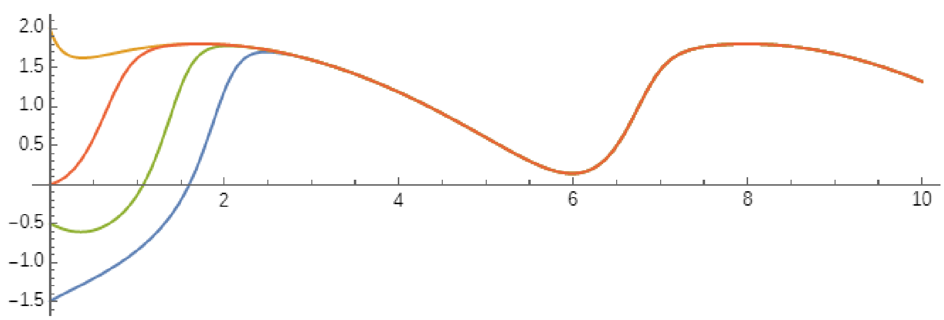
Plot solutions with $x(0)$ as -1.5,-0.5, 0, and 2. What has happened to the "fixed points" from the previous version? Example plot:



Now the fixed "points" have become wavy lines – they depend on time. But there are still two stable *curves* that solutions get attracted to. In fact, if we very carefully play with the initial conditions, we can find that the unstable fixed point is still there too, and we can get there from an initial condition of roughly $x_u = -0.269426825$. Here's a picture showing several other initial conditions near x_u, that slowly peel off towards one side or the other:



(e) Repeat part (d), but now instead of $0.5\sin(t)$ use a larger perturbation of $2\sin(t)$. What has happened to the fixed points now? Is our fixed-point analysis still useful here? Example plot:



Now the perturbation is strong enough that it's destroyed the fixed points completely and merged them into one large attracting curve. Our initial approximate analysis isn't applicable anymore.

2 Coupled equations

Consider the following pair of equations:

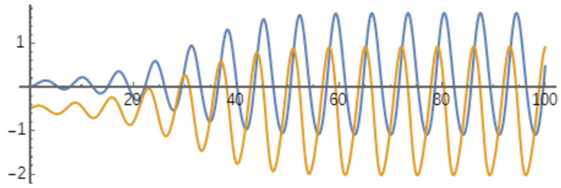
$$x'(t) = 0.5 + y + 0.1y^2 - 0.25x$$

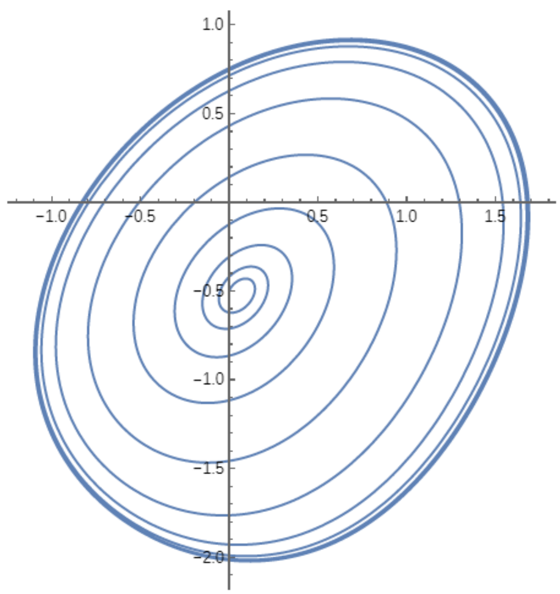
$$y'(t) = 0.3 - x + 0.5y - 0.1y^3$$

(a) Is this time dependent or time independent? Time independent, there's no explicit dependence on t.

(b) Find any fixed points in this system: points where both derivatives $(x'(t), y'(t))$ are zero. Depending on what solver you use, you may find only one solution, or three solutions. If you find three, though, you'll find that only one of them is real (what we care about here). That one solution is $x_0 = 0.0574968$ and $y_0 = -0.511822$

(c) Solve the system from the initial conditions $x(0) = 0$, $y(0) = -0.5$. Solve up until at least $t = 100$. Make one plot with both $x(t)$ and $y(t)$ as functions over time, and another plot showing (x, y) as a parametric curve. Describe the behavior. Example plots:





We see that they initially spiral outwards from their starting point, and then reach a steady cycle. The steady cycle isn't a perfect circle, but has a radius of about 1.5.

(d) Think about the fixed point(s) you found in part (b), and the behavior you observed in part (c). Make a prediction about whether it is stable or unstable, and explain. We know the fixed point was at $(0.06, -0.51)$, and we ran from initial conditions of $(0, -0.5)$ which is pretty close. It spiraled away towards a cycle, so the point appears to be unstable.

(e) Write down the Jacobian matrix, and evaluate it numerically at the fixed point from earlier. Symbolically, the Jacobian is

$$J = \begin{bmatrix} -0.25 & 1 + 0.2y \\ -1 & 0.5 - 0.3y^2 \end{bmatrix}$$

and plugging in our fixed point (x_0, y_0) we get

$$J = \begin{bmatrix} -0.25 & 0.897636 \\ -1 & 0.421411 \end{bmatrix}$$

(f) Using the determinant of the Jacobian, determine if the fixed point is stable or unstable.

$$\det(J) = -0.25 \cdot 0.421411 - 0.897636 \cdot (-1) = 0.792283$$

which is positive, and the trace is

$$\mathrm{Tr}(J) = -0.25 + 0.421411 = 0.171411$$

which is also positive. Therefore the point is unstable. More specifically, J has the two eigenvalues of $0.0857057 \pm 0.885967i$, so it is completely unstable (not a saddle node) and spirals outwards.