ME140A - Homework 3 - Solutions

Due by 11:59PM, Oct 28th, by email to ameiburg@ucsb.edu. Collaboration is encouraged!

1 (Almost) Exponential Decay

Consider a sample of a nuclear isotope decaying over time, $y(x)$. It has a basic rate of decay proportional to its own amount y. But when there's a large quantity, the neutrons it gives off hit more of the sample, accelerating the decay by a factor $1 + y$. We can model this as follows:

$$y'(x) = -(1 + y)y$$

If we have a quantity of 5 units of the substance, this becomes an initial condition

$$y(0) = 5$$

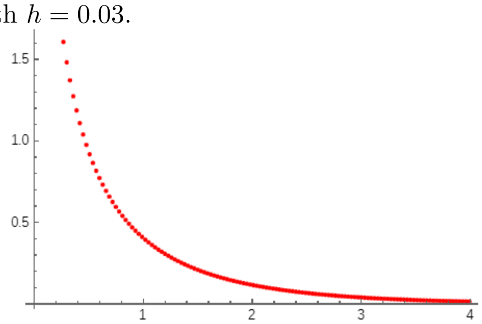
(a) Solve this equation exactly.

$$y(t) = \frac{5}{6e^x - 5}$$

(b) Write MATLAB code to solve this numerically over the interval $x = 0$ to $x = 4$. Use the Euler Method,

$$y_{n+1} = y_n + f(x_n, y_n)h$$

with $h = 0.03$.



(c) Use your data from part **(b)** to estimate when the quantity of the isotope, y, drops to a safe level of 0.04. Then compute the exact time with your equation from **(a)**. How accurate is your estimate?

Exact value: Solving $\frac{5}{6e^x-5} = 0.04$ gives $x = 3.0757$. Our data has $x_{100} = 2.97$, $y_{100} = 0.0401$, and $x_{101} = 3.0$, $y_{101} = 0.0389$. Drawing a line between these, we could say that by $x = 2.975$ the value has dropped to a safe level. Or we could just the first safe point, $x_{101} = 3.0$. This puts us about 0.1 units of time too early.

(d) Run your same code form part **(b)** but with a larger step size of $h = 0.3$. What happens? Explain.

The first step jumps from $y_1 = 5$ down to $y_2 = -4$. This is nonsensical, as we can't have a negative quantity of material. At that point, the equation $-(1+y)y$ stops giving reasonable value, and the "decay" starts accelerating to larger and larger quantities. The values explode and we get quickly get NaNs or negative infinities in our values.

(e) Modify your code from **(b)** to instead use the predictor-corrector method,

$$z = y_n + f(x_n, y_n)h$$

$$y_{n+1} = y_n + \frac{f(x_n, y_n) + f(x_n, z)}{2}h$$

Optimize your code by making sure that you **only use two function evaluations per step**. Use $h = 0.03$.

(f) Again compute the point where $y(x)$ drops below 0.04, with your predictor-corrected method. Compare with part **(c)**. How does the accuracy compare with the Euler Method?

Now we cross over at $x_{103} = 3.06$, $y_{103} = 0.0408$, and $x_{104} = 3.09$, $y_{104} = 0.0395$. Drawing a line between these, we expect it to cross over at $x = 3.0788$. This is much closer to the exact value of 3.0757, with only an error of 0.003 time units – 30x more accurate.

2 Adaptive Step Sizes

Solutions haven't been provided for this, as there is substantial variation in how you can implement this (which two values to compare, how to check error bounds, how aggressively to reduce the step size). You should see step sizes that gradually increase from very small ($h < 0.01$) to much larger ($h = 0.1$) at the end.