ME140A - Homework 2 - Solutions

Due by 11:59PM, Oct 14th, by email to ameiburg@ucsb.edu. Collaboration is encouraged!

1 Problem 1 - Double Integral

Consider the double integral,

$$\int_{x=3}^{4} \int_{y=2}^{3} xy - y^3 + \frac{x}{y} \, dy \, dx$$

1.1 (a)

Compute the exact integral. -6.08087

1.2 (b)

Compute a numerical integral with (the simple, $n = 1$) Simpson's 1/3 rule in each direction. Integral result: -6.080555. This gives a relative error of 5.1×10^{-5}.

2 Problem 2 - Non-rectangular integral

2.1 (a)

How would you integrate an expression like

$$\int_{x=1}^{2} \int_{y=sin(x)}^{x^2} \frac{y^2}{1+e^x} \, dy \, dx$$

You're not expected to write code or evaluate this, just explain the integration approach. Use approach (2) above. When we do the inner integral over y, we'll dynamically choose the bounds to use, depending on our given value of x.

2.2 (b)

How would you integrate $f(x, y)$ over the unit circle, that is, the set of points (x, y) where $x^2 + y^2 \leq 1$? We can rewrite this as

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx$$

and then use the same approach.

3 Problem 3 - Numerical derivative

Define

$$f(x) = \frac{sin(x)}{x^2 + 1}$$

3.1 (a)

Compute the exact derivative of at $x = 0.2$. 0.868899

3.2 (b)

Numerically compute the derivative at $x = 0.2$, using:

1. Forward difference: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$
2. Central difference: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$
3. Five-point stencil: $f'(x) \approx \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$

With $h = 0.1$, $h = 0.01$, and $h = 0.001$. How did the errors compare?

Solution:

```
x = 0.2;
ex = ((x^2+1)cos(x) - 2x*sin(x))/(x^2+1)^2
  0.8688993238136712

for h=[0.1,0.01,0.001]
    fd = (f(x+h)-f(x))/h
    cd = (f(x+h)-f(x-h))/(2h)
    fps = (-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h))/(12h)
    println("h=",h," res = ",[fd,cd,fps]," err = ",[fd-ex,cd-ex,fps-ex])
end

h=0.1 res = [0.8009125296504629, 0.8613724433777378, 0.8687420349141025]
err = [-0.0679867941632083, -0.007526880435933414, -0.00015728889956867498]
h=0.01 res = [0.8626906828866382, 0.8688236653540973, 0.8688993080717554]
err = [-0.006208640927032993, -7.565845957391293e-5, -1.5741915770917103e-8]
```

```
h=0.001 res = [0.8682851809658221, 0.8688985671901212, 0.8688993238121108]
err = [-0.0006141428478491084, -7.566235500355845e-7, -1.5604184611106575e-12]
```

As expected, the forward difference scales as $O(h)$ (each extra tenfold precision in h makes the error ten times smaller). The central difference scales as $O(h^2)$, and the five-point stencil scales as $O(h^4)$. The five-point stencil with $h = 0.1$ already outperforms the forward difference at $h = 0.001$!