1 Midterm 2 - Problem 2 - Worked Solution

The problem statement is:

> **Consider the following system:**
>
> $$x'' = -x - 3x^3 - t^2$$
>
> **with the initial conditions $x(0) = 1$ and $x'(0) = 0$. Use two steps of the predictor-corrector method with $h = 0.5$, to estimate $x(0.5)$ and $x(1.0)$.**

First break the equation down into first-order equations. Use variables x and x'.

$$\frac{d(x)}{dt} = x'$$

$$\frac{d(x')}{dt} = -x - 3x^3 - t^2$$

Here's a worked example of the predictor corrector method from $x(0) = 1$ and $x'(0) = 0$, in other words, the point $y_0 = (1, 0)$. First we compute the derivative at the point y_0, with $t = 0$. The derivative estimates are often referred to by k_i, or y_i', where i goes through the successive estimates.

$$k_1 = y_1' = \left(\frac{d(x)}{dt}, \frac{d(x')}{dt}\right) = (x', -x - 3x^3 - t^2) = (0, -1 - 3 - 0) = (0, -4)$$

Then we make a prediction:

$$y_{pred} = y_0 + h \times y_1' = (1, 0) + (0.5) \times (0, -4) = (1, -2)$$

Then we compute a new derivative at this point y_{pred}, with $t = 0.5$:

$$k_2 = y_2' = (x', -x - 3x^3 - t^2) = (-2, -1 - 3 - 0.5^2) = (-2, -4.25)$$

We average those two derivatives together and use that to get our corrected estimate:

$$y_{corr} = y_0 + h \times \frac{y_1' + y_2'}{2} = (1, 0) + (0.5) \times \frac{(0, -4) + (-2, -4.25)}{2}$$

$$= (1, 0) + 0.5(-1, -4.125) = (0.75, -2.0625)$$

which completes one step. Our value at $t = 0.5$ is $y_1 = y_{corr} = (0.75, -2.0625)$.

In the second step for the problem we don't need to compute $x'(1.0)$, only $x(1.0)$, which means we can save a littttle bit of work. We again compute k_1, now using $y_1 = (0.75, -2.0625)$ and $t = 0.5$:

$$k_1 = (x', -x - 3x^3 - t^2) = (-2.0625, -0.75 - 1.2656 - 0.25) = (-2.0625, -1.1250)$$

and make prediction:

$$y_{pred} = y_0 + h \times k_1 = (0.75, -2.0625) + (0.5) \times (-2.0625, -1.1250) = (-0.5312, -2.6250)$$

This is enough. Now the x' part of k_2 (the first coordinate) is just the x' part of y_{pred} (the second coordinate), so

$$k_2 = (-2.6250, \text{whatever})$$

and the corrected estimate is

$$y_{corr} = y_0 + h \times \frac{k_1 + k_2}{2} = (0.75, -2.0625) + (0.5) \times \frac{(-2.0625, -1.1250) + (-2.6250, \text{whatever})}{2}$$

since we only need the first coordinate, not the "whatever" part,

$$y_{corr} = (-0.6719, \text{whatever})$$

so our final value is $x(1.0) = -0.6719$.

The next page has some MATLAB code to try this out on a computer. You can also download the m-file here.

```
step1 = predictCorrector([1,0], 0, 0.5)
step2 = predictCorrector(step1, 0.5, 0.5)
step2rk3 = rk3(step1, 0.5, 0.5)

%Takes [x,x'] as y, returns their derivatives [x',x'']
function yprime = f(y, t)
    x = y(1);
    xprime = y(2);
    xprime2 = -x - 3*x^3 - t^2;
    yprime = [xprime, xprime2];
end

%Does a step of predictor-corrector, from time t to t+h
function ycorr = predictCorrector(y, t, h)
    k1 = f(y,t);
    ypred = y + h*k1;
    k2 = f(ypred, t+h);
    ycorr = y + h*(k1+k2)/2;
end

%Does a step of 3rd order Runge-Kutte, from time t to t+h
function ycorr = rk3(y, t, h)
    k1 = f(y,t);
    ypred1 = y + h*k1/2;
    k2 = f(ypred1,t+h/2);
    ypred2 = y + h*(2*k2-k1);
    k3 = f(ypred2,t+h);
    ycorr = y + h*(k1+4*k2+k3)/6;
end
```

Running it produces the following output:

```
>> me140a_midterm2_sol

step1 =
    0.5000   -2.0625

step2 =
   -0.6719   -2.3485

step2rk3 =
   -0.6237   -2.2201
```