ME140A - Midterm 1 - Solutions

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1 Integration - 50%

Consider the integral,

$$F = \int_{-1}^{3} x^4 - 2x^3 \, dx$$

Remember that Simpson's 1/3 rule is,

$$\frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

1.1) Compute the exact value of F using algebra.

$$\int_{-1}^{3} x^4 - 2x^3 \, dx = \frac{x^5}{5} - \frac{x^4}{2}\Big]_{-1}^{3} = \frac{243+1}{5} - \frac{81-1}{2} = 48.8 - 40 = \boxed{8.8}$$

1.2) Compute an approximate integral with Simpson's 1/3 rule. Use the *simple* rule, so, $n = 1$ and $h = 1.0$.

$$F \approx \frac{4}{6}(f(-1) + 4f(1) + f(3)) = \frac{4}{6}(3 + 4 \cdot (-1) + 27) = \boxed{\frac{52}{3}}$$

1.3) Compute an approximate integral with the composite Simpson's 1/3 rule, with $n = 2$ and $h = 0.5$.

$$F \approx \frac{4}{12}(f(-1)+4f(0)+2f(1)+4f(2)+f(3)) = \frac{4}{12}(3+4\cdot 0+2\cdot(-1)+4\cdot(0)+27) = \boxed{\frac{28}{3}}$$

1.4) Use Richardson extrapolation, together with your answers from (1.2) and (1.3), to get a more precise estimate. Note that Simpson's 1/3 rule has $O(1/n^4)$ error scaling.

Since error should scale like $1/n^4$, we expect the error of (1.3) to be 16 times smaller than than the error from (1.2). So we estimate the error to be 1/15 of the difference:

$$Err = \frac{1}{15}((\mathbf{1.2}) - (\mathbf{1.3}))$$

Then we subtract that error off our estimate to get the extrapolated answer:

$$F \approx (\mathbf{1.3}) - Err = \frac{16}{15}\frac{28}{3} - \frac{1}{15}\frac{52}{3} = \boxed{8.8}$$

1.5) Compare your answer for (1.1) and (1.4) and give an explanation.

Our answer is *exactly correct!*. But this is no coincidence: the polynomial that we integrate is only 4th order. Simpson's 1/3rd rule is a 2nd-order method, i.e. it fits 2nd order polynomials perfectly. When we extrapolated it, we got a 4th order method, i.e. a new rule that fits 4th order polynomials perfectly. So we would have gotten an exact answer with any 4th-order polynomial we tried to integrate.

2 Differentiation - 50%

You want to use a computer to find the derivative of $f(x) = \sin(x)$ at $x = 1$). The exact answer is, of course, just $\cos(1)$. But you're going to use a finite difference method.

Useful facts: $\sin(1) \approx 0.8414$, $\cos(1) \approx 0.5403$. The Taylor series for $\sin(x)$ at the point $x = 1.0$ is

$$\sin(1+x) = \sin(1) + \cos(1)\,x + \frac{-\sin(1)}{2}\,x^2 + \frac{-\cos(1)}{6}\,x^3 + O(x^4)$$

Do not use a calculator for this problem.

2.1) Write down the formula for a *forward finite-difference* to compute $f'(1)$, with $h = 0.01$. (This uses $f(x+h)$ and $f(x)$.) Don't actually evaluate sin, I just want to see the expression.

$$\boxed{f'(1) \approx \frac{sin(1.01) - sin(1)}{0.01}}$$

2.2) Using the Taylor series for $\sin(1+x)$, estimate how much error your forward difference will have in the derivative, at the given $h = 0.01$.

The main source of error is the second derivative in f.

$$Err = \frac{f(1+h) - f(1)}{h} - f'(1) \approx \frac{(f(1) + f'(1)h + f''(1)h^2/2) - f(1)}{h} - f'(1)$$

$$= \frac{(f(1) - f(1)) + h(f'(1) - f'(1)) + f''(1)h^2/2}{h} = f''(1)h/2$$

Plugging in $f''(1) = -sin(1) = -0.84$ and $h = 0.01$, the error should be about $\boxed{0.0042}$.

2.3) Write down the formula for a *central* finite-difference to compute $f'(1)$, with $h = 0.01$.

Valid answers would be either

$$\boxed{f'(1) \approx \frac{sin(1.01) - sin(0.99)}{0.02}}$$

or

$$f'(1) \approx \frac{sin(1.005) - sin(0.995)}{0.01}$$

depending on your convention of what "h" is - it can be either the distance between the two sample points, or the distance from the middle. We'll use the first formula below in our solutions.

2.4) Using the Taylor series for $\sin(1+x)$, estimate how much error your central difference will have in the derivative, at the given $h = 0.01$.

Similar to problem (2.2), $f(1)$ and $f'(1)$ terms cancel. Expanding to order $O(h^3)$, the error is then

$$\frac{(f''(1)h^2/2 + f'''(1)h^3/6) - (f''(1)(-h)^2/2 + f'''(1)(-h)^3/6)}{h} = \frac{1}{3}f'''(1)h^2$$

Plugging in $f'''(1) = -cos(1) = -0.54$ and $h = 0.01$, the error should be about $\boxed{0.000018}$ or $\boxed{1.8 \times 10^{-5}}$.

2.5) Assume your computer stores 15 decimal digits of precision. This means that, when you add or subtract two numbers x and y, you get an error about $10^{-15}x$ or $10^{-15}y$, whichever is bigger. Write down a expression for the error from the *central* finite-difference from (2.3), that depends on h. This expression should include both the precision error from your computer, and the error from the finite-difference method itself. *There will be some range of answers will be accepted, because of different ways to estimate the rounding error.*

When we evaluate $f(1.01)$ and $f(0.99)$, we get a relative error of 10^{-15}. Since $f(1)$ is on the scale of 1 itself, the relative error means we also get an absolute error of 10^{-15}. We add these two together, and then divide by h. Dividing by h makes the error much bigger, and we can expect it to add about

$$\frac{2 \times 10^{-15}}{h}$$

to the overall error. Adding this to our expression from (2.4), we can estimate that our final error is

$$\frac{1}{3}f'''(1)h^2 + \frac{2 \times 10^{-15}}{h}$$

Plugging in a value of $|f'''(1)| = 0.54$, we estimate

$$\boxed{0.18 \times h^2 + \frac{2 \times 10^{-15}}{h}}$$

The factor of "2" in the rounding error could plausibly be a few times bigger or smaller. We could say there's rounding error from each sin, and some rounding error from the subtraction as well. There could also be rounding error in the $1 + h$ itself. That's 2, 3, or 4 rounding errors. We could assume all the errors compound (so that it's 2, 3 or 4 times 10^{-15}) or treat them as uncorrelated (so that it's $\sqrt{2}$, $\sqrt{3}$, or $\sqrt{4}$ times 10^{-15}).

2.6) Using your error formula from (2.5), figure out an optimal value of h to minimize your total error. *Any answer within a factor of 4 of the optimum will be accepted, because of different ways to estimate the rounding error in (2.5).*

One easy and popular heuristic is to pick h so that the two sources of error about equal in size. So we solve

$$0.18 \times h^2 = \frac{2 \times 10^{-15}}{h} \implies h^3 = 11 \times 10^{-15}$$

$$\implies h = 1.1 \times 10^{-14} \approx \boxed{0.000022 = 2.2 \times 10^{-5}}.$$

Another approach would be to minimize our expression in 2.5), by taking the derivative and setting that to zero. That solves to

$$h^3 = 5 \times 10^{-15} \implies h \approx 1.7 \times 10^{-5}$$

which we see is pretty close.t