ME140A - Homework 5

Due by 11:59PM, Monday Nov 21st, by email to ameiburg@ucsb.edu.
Collaboration is encouraged!

1 Plotting a Poincaré Map

The following equation of motion is the so-called Duffing Oscillator:

$$x'' = x - x^3 - 0.02x' + 3\sin(t)$$

(a) Rewrite the Duffing oscillator as two coupled first-order differential equations.

(b) Plot the solution for $x(0) = 0$ and $x'(0) = 0$. (You can show either both functions on the same graph, or an (x, x') parametric plot).

(c) Looking at your graph for part (b), do you see any fixed points or limit cycles? Explain.

(d) Because we have a time dependence, we can't blindly apply fixed point analysis. But, the time dependence is periodic with period 2π. If we solve the system from initial conditions $(x(0), x'(0)) = (x_0, v_0)$, then we have a *mapped point* from our solution $(x(2\pi), x'(2\pi))$. Write a MATLAB function to compute this map: it should take two values (x_0, v_0) and give the resulting values $(x(2\pi), x'(2\pi))$. This function is called the Poincaré map.

(e) If we had a different set of initial conditions that eventually evolves to $(x(n2\pi), x'(n2\pi)) = (x_0, v_0)$, what can say about the solution at $(x((n+1)2\pi), x'((n+1)2\pi))$? Explain why this Poincaré map might be useful, and what it can tell us about the system.

(f) Given the initial conditions $(0, 0)$, repeatedly apply the Poincaré map to get a new point, and then apply it again to that point, and so on. Do this $N = 50000$ times. Display these points as dots on a graph. Don't connect them, make sure to make a scatter plot! You should get something pretty amazing!

A note: you can see that your points don't go *randomly* on the graph; they omit certain regions, and define a certain set. If you start somewhere else, your Poincaré map will pull you towards this cloud. So, this set of states an *attractor*, just like a stable fixed point or a stable limit cycle. On the other hand, it has an infinitely complicated shape, and an infinitely complicated set of dynamics. (Compared with a limit cycle, which has a simple shape, and a simple rule for moving: just go around it!) Since this shape is so odd, it's called a *strange attractor*.

(g) Just for this part, change your equation to read

$$x'' = x - x^3 - 0.01x' + 0.5\sin(t)$$

Play around with this Poincare map until you see what it does. It's much better behaved than the previous values. For the underlying ODE, what period in t does this system approach?

2 Finding Chaos

(Don't forget to change the numbers back from part (g)!)

Now we would like to observe how the long-term results change depending on a tiny starting change. Start from an initial point $(0.01\sin(\theta), 0.01\cos(\theta))$, where θ takes on 20 equally spaces values from 0 to 2π. This little circle of 20 points will be our "test region", and they represent some uncertainty in our initial conditions $(0, 0)$. We want to see how the region moves together.

(a) Plot the curve connecting the points in our test region. It should be a small circle around $(1, 1)$. Then, apply the Poincaré map to each of the points, and plot the new shape. It should still be approximately circular.

(b) Now apply the map to each of your points, three more times (for 4 mappings total). Plot it on a new graph. What do you notice about the size of the region? What about the shape?

(c) Apply the map to each of your points, four more time (for 8 mappings total). What does it look like now?

(d) Finally, apply the map to each of your points, two more times (for 10 mappings total). What does it look like now? If your initial estimates had this error of 0.01, would you trust this simulation for any engineering work? Roughly how much time (in t) did it take for the results to fall apart?

This kind of extreme sensitivity to initial conditions is the curse of *chaos*. It can only happen when we have a periodically *driven* system (like the above), or when we have three or more variables. The textbook example of chaos in a system of three first-order ODEs is the Lorenz system (link! Go check out the animations on Wikipedia). It was found while trying to make a simplified model of weather.

For a bit of extra credit, show that the Duffing oscillator we studied above can be written as a system of *four* first-order ODEs, with no time-dependence. That is, find an equivalent system:

$$w'(t) = f_1(w, x, y, z)$$

$$x'(t) = f_2(w, x, y, z)$$

$$y'(t) = f_3(w, x, y, z)$$

$$z'(t) = f_4(w, x, y, z)$$

and how to map some Duffing oscillator initial conditions into this four-variable system.