ME140A - Homework 4

Due by 11:59PM, Monday Nov 7th, by email to ameiburg@ucsb.edu.
Collaboration is encouraged!

1 1D Fixed Points

Fixed point analysis is often a useful way to understand the dynamics of a time-independent differential equation. Even if an equation has time dependence, as long as this isn't too big, we can try to apply fixed point analysis anyway. This will walk through one such scenario.

(a) Using an differential equation solver – it could be your own from last homework, or one of MATLAB's built in solvers, like "dsolve" – solve the following time independent system:

$$x'(t) = 0.3 - x^3 + 2x$$

Plot the solutions with the following initial values for $x(0)$: -3, -2, -1, -0.5, -0.25, 0, 1, 2, and 3. Show at least the behavior from $t = 0$ up to $t = 10$.

(b) Remember that a *fixed point* of a system is where all the derivatives are zero. Using algebra (and not a differential equation solver), identify all three fixed points in the above equation. How does this match up with your observed behavior?

(c) Given a fixed point (t_0, x_0) where $x'(t) = f(x) = 0$, we say the fixed point is *stable* if $\frac{\partial f}{\partial x} < 0$, and *unstable* if $\frac{\partial f}{\partial x} > 0$. (When it $= 0$, it can still be stable or unstable, but it's harder to determine.) Classify the points from part (b) as stable or unstable. How does this match up with your observed behavior?

(d) Suppose that we now add a *time-dependent* perturbation to the system:

$$x'(t) = 0.3 - x^3 + 2 * x + 0.5\sin(t)$$

Plot solutions with $x(0)$ as -1.5,-0.5, 0, and 2. What has happened to the "fixed points" from the previous version?

They're no longer truly fixed points, because we have time dependence. But we see that studying the *homogeneous* version of the system (with the sin term removed) helps us understand the behavior of the time dependent equation as well.

(e) Repeat part (d), but now instead of $0.5\sin(t)$ use a larger perturbation of $2\sin(t)$. What has happened to the fixed points now? Is our fixed-point analysis still useful here?

2 Coupled equations

Consider the following pair of equations:

$$x'(t) = 0.5 + y + 0.1y^2 - 0.25x$$

$$y'(t) = 0.3 - x + 0.5y - 0.1y^3$$

(a) Is this time dependent or time independent?

(b) Find any fixed points in this system: points where both derivatives $(x'(t), y'(t))$ are zero. You can use Wolfram Alpha to solve the resulting (not-differential) equations.

(c) Solve the system from the initial conditions $x(0) = 0$, $y(0) = -0.5$. Solve up until at least $t = 100$. Make one plot with both $x(t)$ and $y(t)$ as functions over time, and another plot showing (x, y) as a parametric curve. Describe the behavior. You can use MATLAB's solver or your own.

(d) Think about the fixed point(s) you found in part (b), and the behavior you observed in part (c). Make a prediction about whether it is stable or unstable, and explain.

(e) In problem 1, we said that a point is stable if $\frac{\partial f}{\partial x} < 0$. This works for 1-variable problems, but now we have two variables, where the condition is more complicated. If

$$x'(t) = f(x, y), \quad y'(t) = g(x, y)$$

then define the Jacobian matrix as

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Write down the Jacobian matrix, and evaluate it numerically at the fixed point from earlier.

(f) The determinant of the matrix $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined as $\det(J) = ad - bc$. Compute the determinant of the Jacobian from part (e). If $\det(J) > 0$, the point is unstable, and if $\det(J) < 0$, the point is stable. Which is it?