ME140A - Homework 1

October 6, 2022

Due by 11:59PM, Oct 6th, by email to ameiburg@ucsb.edu. Collaboration is encouraged!

1 Problem 1 - Precision

1.1 (a)

This course will mostly focus on *algorithmic* or *truncation* error, that is, errors from inexact algebra. ($\sqrt{1+x} \approx 1 + \frac{x}{2}$, for instance.) Errors from the intrinsic precision limits of computers can be relevant as well, however.

The following expression simplifies to 1:

$$f(x, y) = \frac{(x+y)^2 - 2xy - y^2}{x^2}$$

Write this expression as a function (in MATLAB, or the language of your choice) and evaluate at the points $(x, y) = (10^m, 10^n)$ for m = -2 to -7, and n = +3 to +6. Some of the values will be very close to 1.0, and some will be farther away. Explain why you think this is, and note how the accuracy seems to depend on m and n.

1.2 (b)

The $exp(x)$ function evaluates the exponential e^x. Evaluate the expressions,

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exp(500) + (-exp(500) + exp(1))
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and

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(exp(500) + -exp(500)) + exp(1)
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and explain the difference.

We see that addition on the computer isn't *commutative*: $a + (b + c) \neq (a + b) + c$. This can become important when taking very long sums. Suppose we are computing the sum of a long, sorted array full of small positive numbers. The accuracy of our answer can depend on whether we start at the "small" end or the "large" end of the array. Which do you think would be more precise?

2 Problem 2 - Taylor Series

Suppose we are given two functions in terms of their second-order Taylor series,

$$f(x) = 1 + \frac{3}{2}x - \frac{1}{10}x^2 + O(x^3)$$

$$g(x) = 0 + 4x - 5x^2 + O(x^3)$$

Compute the Taylor series for $f(g(x))$ and $g(f(x))$. You can use either chain rules for derivatives (e.g. $\frac{d(f(g(x)))}{dx} = f'(g(x))g'(x)$) or by directly combining the polynomials.

3 Problem 3 - Implementing an Integration Rule

The next-order rule after Simpson's rule is Boole's Rule,

$$\int_a^b f(x)\,dx = \frac{2(b-a)}{45}\left(7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{2a+2b}{4}\right) + 32f\left(\frac{3a+b}{4}\right) + 7f(b)\right)$$

Implement this in MATLAB and use it to evaluate $\int_0^{1.8} \sin(e^x)$ until you have 5 digits of precision (a 10^{-5} relative error). Graph the function and comment briefly on how your integration converges.

4 Problem 4 - Undetermined Coefficients

We discussed how to use Gauss quadrature – choosing the locations of sample points – to get zero error on certain functions. The trapezoidal rule has zero error on linear functions, but has error on quadratic functions; the two-point Gauss-Legendre formula also uses only two samples, but has zero error on quadratic functions.

We don't have to only focus on polynomials, though. Suppose we want a two-point integration rule that computes the following four integrals exactly:

$$\int_{-1}^{1} 1\,dx, \qquad \int_{-1}^{1} e^x\,dx$$

$$\int_{-1}^{1} e^{2x}\,dx, \qquad \int_{-1}^{1} e^{3x}\,dx$$

where our integration rule is of the form $I = \int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$. Write down the equations for your four variables (c_1, x_1, c_2, x_2) and use a computer to find the solution.