ME140A - Final - Open Book

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1 Numerical Differentiation - 20%

(a) Consider the following three differentiation rules (all are standard ones we talked about in class, and also in your book):

$$(1): \qquad f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$(2): \qquad f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$(3): \qquad f'(x) \approx \frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)}{12h}$$

For each one, first state the error scaling in h. For each one, give a scenario where you would want to pick that method over the other two, and explain why.

(b) I have a function f, and I want to evaluate f' at a certain point x. I know that $f(x)$ is roughly 10^5, that $f'(x)$ is roughly 10^2, and my computer has a relative precision of roughly 10^{-15}. (This relative precision means that if I store a number of size 10^{10}, the error will be about $10^{-15} \times 10^{10}$, so my number will be $10^{10} \pm 10^{-5}$.) I'm going to use differentiation rule (1) above. What value of h should I pick (roughly) to maximize my accuracy? How much accuracy should I expect in the result?

2 Numerical Integration - 25%

(a) We learned about several numerical integration methods. The simplest is the left sum, where we break the interval $[a, b]$ into n intervals $[x_i, x_{i+1}]$ and say

$$\int_a^b f(x) \approx \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

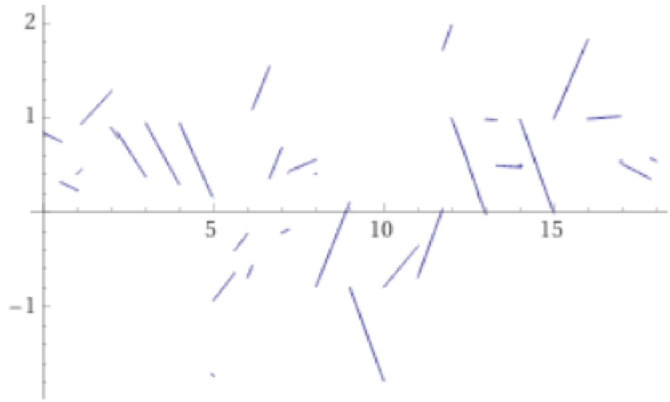
The trapezoid method is:

$$\int_a^b f(x) \approx \frac{b-a}{n} \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i+1})}{2}$$

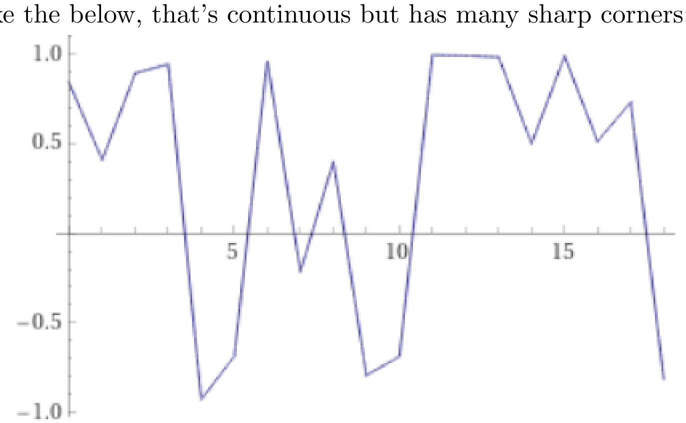
And Simpson's method is:

$$\int_a^b f(x) \approx \frac{b-a}{n}\sum_{i=1}^{n}\frac{f(x_i)+4f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})}{6}$$

Each one is a "higher-order" method than the previous. Now imagine that we want to integrate a discontinuous function like the one plotted below:



Discuss the error scaling of the three formulas on a function like the above. When we talk about the order of an integration method, what kinds of assumptions are we making? How does that work out here? What about for a function like the below, that's continuous but has many sharp corners:



(b) I have a function $f(x, y)$ that I want to integrate on the range $x = [6, 8]$, $y = [1, 9]$. I'm going to use Simpson's Rule. I will use $n_x = 1$ (no dividing into smaller intervals along x) and $n_y = 2$ (dividing into two smaller intervals). Write the expression for how to evaluate the integral, in terms of which places to evaluate f.

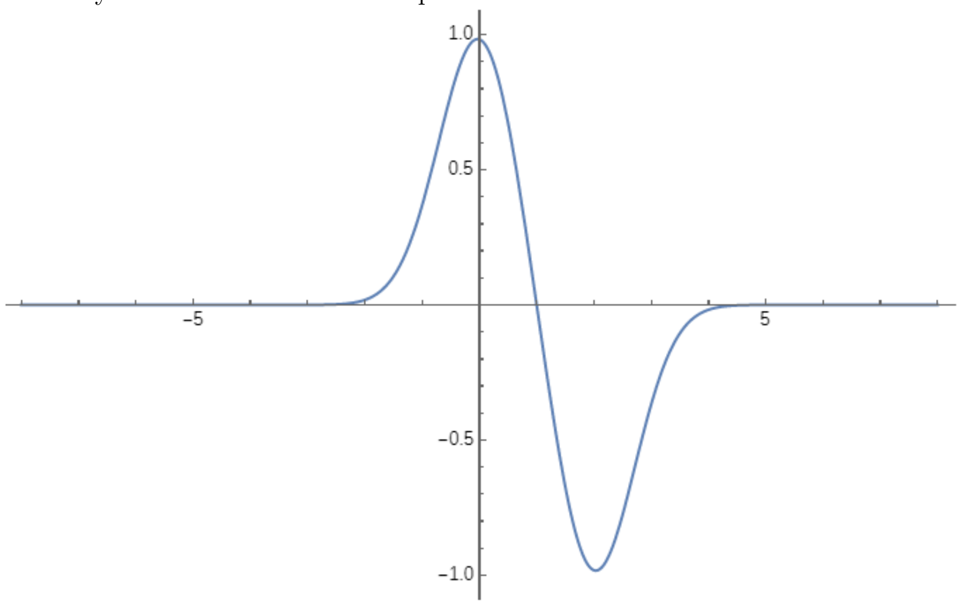
Your answer should be an expression in a form like $\frac{1}{4}f(0, 3) + \frac{1}{6}f(10, 3.5) + \frac{2}{7}f(1, 1)$ or similar. The number of f's be between 10 and 20, but I won't say the particular number.

(c) You are trying to evaluate the following integral:

$$\int_{-8}^{8} e^{-x^2} - e^{-(x-2)^2}$$

You know based on theory the answer should be something on the scale of 10^{-17}, but when you use MATLAB's numerical integration library, you're getting garbage values like -2.3×10^{-12} or 4.1×10^{-13}. Explain what's going wrong with the numerical integration, and give a simple change you could make to make the software perform better.

You may find it useful to look at a plot of the function:



3 Euler Method - 25%

Just to remind you – the Euler method is the simplest method of solving initial value problems that we learned. For a first order problem

$$x'(t) = f(x, t)$$

the Euler method repeatedly computes

$$x_{k+1} = f(x_k, t_k)$$

(a) Explain how you would solve the problem

$$x''' - 2x'' + \frac{x}{t} = 0$$

$$x(1) = 1, \quad x'(1) = 0, \quad x''(1) = 0$$

with the Euler method. You don't have to actually numerically evaluate anything, but write down the expressions you would need for the first steps, defining whatever variables you need.

(b) Consider the following initial value problem:

$$x'' = -(3 + 2.5\sin(t/5))^2 x + \cos(t)$$

$$x(0) = 1, \quad x'(0) = 2$$

You can physically interpret this as follows. $x'' = -k^2 x$ is a harmonic oscillator with frequency of k, so this is a harmonic oscillator with frequency that varies between $3 - 2.5 = 0.5$ and $3 + 2.5 = 5.5$. It's driven by an external force of $\cos(t)$.

You're going to solve this problem to compute $x(100)$ with an accuracy of 10^{-3}. You will use the Euler method, and each timestep will be size $100/n$, so that you take n steps total. As a rough estimate, how large will n need to be get the desired accuracy? You don't need to think about the exact dynamics of this system – that's what the computer solver is for! – but you should think about roughly how this system behaves in order to inform your estimate for n.

(As a note: this system is *linear*, which is enough to guarantee that there isn't any chaos. So you *can* strive for such good accuracy! As you saw in the chaos homework, when the system is chaotic, there's no hope of keeping accuracy out to $t = 100$.)

4 Fixed Points - 20%

(a) Find all fixed points of the third-order system,

$$x''' = \frac{x'' + x'}{5 + x^2} + x^2 x' + \frac{sin(x'')}{\cos(x + x') + 2} + \sin(x - 17x' + 5)$$

(b) Find all fixed points of

$$x' = -4 - 4x - 31x^2 - 10x^3 + y^2$$

$$y' = y - x^2 - 4x - 2$$

(They should all be very simple numbers.) Classify each fixed point as stable, unstable, saddle node, and whether they spiral or not.

(c) If we "reverse time", we get the mirror system

$$x' = +4 + 4x + 31x^2 + 10x^3 - y^2$$

$$y' = -y + x^2 + 4x + 2$$

where we've just negated each derivative. Based on your information from part (b), without solving the system from scratch, classify the fixed points of the new system. Explain the connection between fixed points of the original system and the mirror system by drawing at least one picture.

5 Boundary Value Problems - 10%

(a) The shooting method relies on an IVP solver underneath. When using the shooting method for solving a boundary value problem, you might want to decrease the step size h of the IVP underneath, as you keep improving your "shot". Explain why you might want to do this, and give a suggestion for how a shooting point algorithm could detect when it needs to decrease the step size.

(b) You're need to solve the boundary value problem for a ball flying through the air on a ballistic trajectory, solving for $x(t)$ and $y(t)$. It's not a simple parabola, because you're including a very precise model for air resistance that depends on x' and y' (air resistance depends on speed and direction) and also on y (because at higher altitudes the air gets thinner). So you have a boundary value problem like

$$x''(t) = f_x(x', y', y)$$

$$y''(t) = f_y(x', y', y) - \text{gravity}$$

$$x(0) = 0,\ y(0) = 0,\ x(30) = 75,\ y(30) = 0$$

where f_x and f_y is your model for air resistance.

But! A parabola is certainly a very good approximate solution. Explain how you could use your knowledge of parabolic trajectories with your boundary value problem solver, for it to solve the problem more accurately and reliably. Which of the two types of BVP solvers could make better use of this information?